Associative Bilinear Forms in Some Baric Algebras*

R. Baeza-Vega and R. Benavides
Departamento de Matemática y Estadística
Casilla 54-D
Temuco, Chile

Submitted by Richard A. Brualdi

ABSTRACT

Let $(\mathbf{A}, \omega)$ be a K-algebra, $\mathbf{e} \in \mathbf{A}$ a nontrivial idempotent, K an infinite field whose characteristic is different from 2, and $\mathbf{A} = K\mathbf{e} \oplus U \oplus V$, where $U = \{x \in \ker \omega | \mathbf{e}x = \frac{1}{2}x\}$. If $\mathfrak{B} : \mathbf{A} \times \mathbf{A} \to K$ is an associative bilinear form, we prove the two following main facts: (1) If $\mathbf{A}$ is a kth order Bernstein algebra and $\mathfrak{B}$ is a nondegenerate form, then $\mathbf{A}$ is a kth order quasiconstant algebra and the idempotent is unique. (2) If $x^3 - (1+\gamma)\omega(x)x^2 + \gamma\omega(x)^2 x = 0$ is the rank equation of $\mathbf{A}$ and if $\mathfrak{B}$ is nondegenerate, then $\mathbf{A}$ is a Jordan algebra. If $0 \neq \gamma \neq 1$, then $\mathfrak{B}$ is degenerate, the symmetry of $\mathfrak{B}$ only depends on the symmetry of $\mathfrak{B}|_{\mathbf{V}}$, and $\mathfrak{B}(\mathbf{e}_1, \mathbf{e}_1) = \mathfrak{B}(\mathbf{e}, \mathbf{e})$ for any idempotent elements $\mathbf{e}$, $\mathbf{e}_1$ in $\mathbf{A}$.

1. PRELIMINARIES

Throughout this paper $(\mathbf{A}, \omega)$ will be a K-baric algebra and K an infinite field whose characteristic is different from two. That means $\mathbf{A}$ is a commutative nonassociative algebra over K, and $\omega : \mathbf{A} \to K$ a nontrivial algebra homorphism. Let $\mathbf{e} \in \mathbf{A}$ be fixed such that $\omega(\mathbf{e}) = 1$, and let N denote the kernel of ω. Then we have $\mathbf{A} = K\mathbf{e} \oplus N$. Let $\tau : N \to N$ denote the left multiplication by $\mathbf{e}$ in N. In Sections 2 and 3 below, we highlight the

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existence of a vector subspace U in A such that $U = \{x \in N | ex = \frac{1}{2}x\}$, and we write

$$\mathbf{A} = K\mathbf{e} \oplus U \oplus V, \tag{1}$$

where V is a complementary subspace of U in N.

Let $\mathfrak{B} : \mathbf{A} \times \mathbf{A} \to K$ be a bilinear form. $\mathfrak{B}$ is associative when the identity $\mathfrak{B}(xy, z) = \mathfrak{B}(x, yz)\ \forall x, y, z \in \mathbf{A}$ holds. $\mathfrak{B}$ is a symmetric form if $\mathfrak{B}(x, y) = \mathfrak{B}(y, x)\ \forall x, y \in \mathbf{A}$. We say that $\mathfrak{B}$ is a degenerate form if $x \in \mathbf{A}$ exists such that $\mathfrak{B}(x, y) = 0$ or $\mathfrak{B}(y, x) = 0\ \forall y \in \mathbf{A}$.

Using the decomposition (1), for an associative bilinear form, we have $\mathfrak{B}(\mathbf{e}, n) = \mathfrak{B}(\mathbf{e}^2, n) = \mathfrak{B}(\mathbf{e}, \mathbf{e}n) = \mathfrak{B}(\mathbf{e}n, \mathbf{e}) = \mathfrak{B}(n, \mathbf{e}^2) = \mathfrak{B}(n, \mathbf{e})$, $n \in N$, so

$$\mathfrak{B}(\mathbf{e}, N) = \mathfrak{B}(N, \mathbf{e}). \tag{2}$$

We also have $\mathfrak{B}(\mathbf{e}, u) = \mathfrak{B}(\mathbf{e}^2, u) = \mathfrak{B}(\mathbf{e}, \mathbf{e}u) = \mathfrak{B}(\mathbf{e}, \frac{1}{2}u) = \frac{1}{2}\mathfrak{B}(\mathbf{e}, u)$, $u \in U$, so

$$\mathfrak{B}(\mathbf{e}, U) = \mathfrak{B}(U, \mathbf{e}) = 0. \tag{3}$$

From (1), (2), and (3) we obtain the matrix canonical block decomposition of $\mathfrak{B}$ asxxx

$$[\mathfrak{B}] = \begin{array}{c} \begin{array}{ccc} K\mathbf{e} & U & V \end{array} \\ \begin{pmatrix} \mathfrak{B}(\mathbf{e}, \mathbf{e}) & 0 & * \\ 0 & * & * \\ * & * & * \end{pmatrix} \end{array} \begin{array}{c} K\mathbf{e} \\ U \\ V \end{array} \tag{4}$$

We will study (4) to find those kth Bernstein algebras and T-algebras of rank 3 which have a nondegenerate associative bilinear form.

2. kTH ORDER BERNSTEIN ALGEBRAS

We define the plenary powers of $x \in A$ by $x^{[1]} := x$ and $x^{[i+1]} := x^{[i]}x^{[i]}$ for every integer $i \geqslant 1$. The baric algebra $(\mathbf{A}, \omega)$ is a kth order Bernstein algebra if any element $x \in \mathbf{A}$ satisfies the identity $x^{[k+2]} = \omega(x)^{2^k} x^{[k+1]}$. Naturally k is the smallest integer such that this identity holds.

It is well known that every kth order Bernstein algebra has an idempotent $\mathbf{e}$ such that $\omega(\mathbf{e}) = 1$. The proposition established by Hentzel, Peresi, and Holgate in [3] allows us to write the decomposition $N = U \oplus V$, where

$U = \{n \in N | \mathbf{e}n = \frac{1}{2}n\}$ and $V = \ker \tau^k$ with $U^2 \subseteq V$. Let W be a complementary subspace to U^2 in V, so that $U^2 \oplus W = V$.

For any associative bilinear form $\mathfrak{B}$ in $\mathbf{A}$, we now perform some calculations:

$$\mathfrak{B}(\mathbf{e}, v) = \mathfrak{B}(\mathbf{e}^{k+1}, v) = \mathfrak{B}(\mathbf{e}^k, \tau v) = \mathfrak{B}(\mathbf{e}^{k-1}, \tau^2 v) = \cdots$$

$$= \mathfrak{B}(\mathbf{e}, \tau^k v) = \mathfrak{B}(\mathbf{e}, 0) = 0$$

for every $v \in V$. Similarly $\mathfrak{B}(v, \mathbf{e}) = 0 \ \forall v \in V$. So $\mathfrak{B}(\mathbf{e}, V) = \mathfrak{B}(V, \mathbf{e}) = 0$, and we obtain, by (3),

$$\mathfrak{B}(\mathbf{e}, N) = \mathfrak{B}(N, \mathbf{e}) = 0. \tag{5}$$

For $u \in U$ and $n \in N$, we have $\mathfrak{B}(u, n) = 2\mathfrak{B}(\frac{1}{2}u, n) = 2\mathfrak{B}(\mathbf{e}u, n) = 2\mathfrak{B}(\mathbf{e}, un) = 0$ because $un \in N$ and by (5). Similarly $\mathfrak{B}(n, u) = 0$. Based on the identity (3), it follows that

$$\mathfrak{B}(U, \mathbf{A}) = \mathfrak{B}(\mathbf{A}, U). \tag{6}$$

For each generator $u_i u_j$ of U^2 and $n \in N$, we have $\mathfrak{B}(u_i u_j, n) = \mathfrak{B}(u_i, u_j n) = 0$, and similarly $\mathfrak{B}(n, u_i u_j) = 0$ by (6). Then

$$\mathfrak{B}(U^2, N) = \mathfrak{B}(N, U^2) = 0. \tag{7}$$

Hence, the matrix canonical block decomposition of $\mathfrak{B}$ is

$$[\mathfrak{B}] = \begin{array}{cc} \begin{array}{cccc} K\mathbf{e} & U & U^2 & W \end{array} & \\ \begin{pmatrix} \mathfrak{B}(\mathbf{e}, \mathbf{e}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \end{pmatrix} & \begin{array}{c} K\mathbf{e} \\ U \\ U^2 \\ W \end{array} \end{array} \tag{8}$$

This matrix form, valid for every dimension of A, has the following consequences:

PROPOSITION 1. *Any associative bilinear form $\mathfrak{B}$ on the kth order Bernstein algebra $\mathbf{A}$ is totally determined by its action on $K\mathbf{e} \oplus W$.*

Proof. It is obvious, because if $x = \alpha\mathbf{e} + u + v_1 + v_2$, $y = \beta\mathbf{e} + u'$

$+ v_1' + v_2'$, with $\alpha, \beta \in K$, $u, u' \in U$, $v_1, v_1' \in U^2$, $v_2, v_2' \in W$, then $\mathfrak{B}(x, y) = \alpha\beta\,\mathfrak{B}(\mathbf{e}, \mathbf{e}) + \mathfrak{B}(v_2, v_2')$. ∎

PROPOSITION 2. *If there exists a nondegenerate associative bilinear form* $\mathfrak{B}$ *on the kth order Bernstein algebra* $\mathbf{A}$, *then* $x^{[k+1]} = \omega(x)^{2^k}\mathbf{e}\ \forall x \in \mathbf{A}$, *and the idempotent* $\mathbf{e}$ *is unique.*

Proof. If $\mathfrak{B}$ is nondegenerate, $U = \{0\}$. Then the result follows from Theorem 5.10 of C. Mallol [4]. ∎

3. *T*-ALGEBRAS OF RANK 3

The K-baric algebra $(\mathbf{A}, \omega)$ is a T-algebra if there exist fixed elements $\gamma_1, \dots, \gamma_{r-1}$ in K such that the identity $x^r + \gamma_1\omega(x)x^{r-1} + \cdots + \gamma_{r-1}\omega(x)^{r-1}x = 0$ holds in $\mathbf{A}$. The unique equation of this type with minimum degree is called the rank equation of $\mathbf{A}$, and r is the rank of $\mathbf{A}$. When the rank of a T-algebra $(\mathbf{A}, \omega)$ is 3, the rank equation is

$$x^3 - (1 + \gamma)\omega(x)x^2 + \gamma\omega(x)^2 x = 0, \tag{9}$$

because the identity $1 + \gamma_1 + \cdots + \gamma_{r-1} = 0$ holds in every T-algebra. In this work we will only consider T-algebras of rank 3.

LEMMA. *If the T-algebra satisfies* $\omega(x)[x^2 - \omega(x)x] = 0\ \forall x \in \mathbf{A}$, *then* $x^2 - \omega(x)x = 0$ *holds in* $\mathbf{A}$.

Proof. Let us observe that $\forall x \notin \ker\omega$ we have $x^2 - \omega(x)x = 0$. Furthermore, if $x \in \ker\omega$ with $x^2 - \omega(x)x \neq 0$, then $x^2 \neq 0$. For any $y \notin \ker\omega$ we have $\omega(x + \lambda y)[(x + \lambda y)^2 - \omega(x + \lambda y)(x + \lambda y)] = 0\ \forall\lambda \in K$; therefore $\omega(y)x^2 = 0$, but $\omega(y) \neq 0$, so $x^2 = 0$. This fact contradicts the hypothesis. Thus $x^2 - \omega(x)x = 0\ \forall x \in A$. ∎

PROPOSITION 3. *Let* $(\mathbf{A}, \omega)$ *be a T-algebra of rank* 3. *Then* $\mathbf{A}$ *is a Jordan algebra if and only if* $\gamma = 0$ *or* $\gamma = 1$.

Proof. $\Leftarrow$: If $\gamma = 0$ or $\gamma = 1$, we have $x^3 - \omega(x)x^2 = 0$ or $x^3 - 2\omega(x)x^2 + \omega(x)^2 x = 0$, so $\mathbf{A}$ is a Jordan algebra; see [5, Theorem 2.3.2].

$\Rightarrow$: Let $\mathbf{A}$ be a Jordan algebra. From [5, Theorem 2.3.2], the rank equation is $x^3 - \omega(x)x^2 = 0$ or $x^3 - 2\omega(x)x^2 + \omega(x)^2 x = 0$. First, we suppose that the equation is $x^3 - \omega(x)x^2 = 0$. Now the difference between this and the identity (9) is $\gamma\omega(x)[x^2 - \omega(x)x] = 0$.

If $\gamma \neq 0$ then $\omega(x)[x^2 - \omega(x)x] = 0$ $\forall x \in A$, and from the above Lemma we obtain $x^2 - \omega(x)x = 0$. This contradicts the rank of $(\mathbf{A}, \omega)$, giving $\gamma = 0$.

An analogous argument proves the second case, i.e. that when the rank equation is $x^3 - 2\omega(x)x^2 + \omega(x)^2x = 0$ we have $\gamma = 1$. ■

The T-algebras originated from modeling genetic problems. In this case, the existence of an idempotent element is frequent. For this reason, from now on we will suppose that every T-algebra $(\mathbf{A}, \omega)$ has an idempotent element of weight 1. It is well known that for every idempotent element $\mathbf{e}$ of weight 1 we can write $\mathbf{A} = K\mathbf{e} \oplus N$, where $N = \ker \omega$. Let $x = \mathbf{e} + \lambda n$ with $\lambda \in K$ and $n \in N$. Then from (9) we obtain

$$\gamma n + 2\mathbf{e}(\mathbf{e}n) - (1 + 2\gamma)\mathbf{e}n = 0, \tag{10}$$

$$\mathbf{e}n^2 + 2n(\mathbf{e}n) = (1 + \gamma)n^2. \tag{11}$$

Rewriting (10), we deduce $\mathbf{e}(n - 2\mathbf{e}n) = \gamma(n - 2\mathbf{e}n)$ and $\mathbf{e}(\mathbf{e}n - \gamma n) = \frac{1}{2}(\mathbf{e}n - \gamma n)$. If $U := \{n \in N | \mathbf{e}n = \frac{1}{2}n\}$ and $V := \{n \in N | \mathbf{e}n = \gamma n\}$, we obtain the identities $\mathbf{e}n - \gamma n \in \ker(\tau - \frac{1}{2}I) = U$ and $n - 2\mathbf{e}n \in \ker(\tau - \gamma I) = V$. Then we have

$$\left(\tfrac{1}{2} - \gamma\right)n = (\mathbf{e}n - \gamma n) + \tfrac{1}{2}(n - 2\mathbf{e}n) \in U + V. \tag{12}$$

If $\gamma \neq \frac{1}{2}$, N can be written as the direct sum of the two subspaces, and consequently (1) holds. Linearizing the identity (11), we obtain the following relations:

$$U^2 \subseteq V, \qquad UV \subseteq U, \quad \text{and} \quad V^2 = 0. \tag{13}$$

We remark that in this case, the conditions for (4) have been satisfied, i.e. $\mathfrak{B}(\mathbf{e}, U) = \mathfrak{B}(U, \mathbf{e}) = 0$.

With the notation of the Lemma in [1], we can write $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \gamma \neq \frac{1}{2}$, $N_1 = U$, and $N_2 = V$. So we find $\mathfrak{B}(U, V) = \mathfrak{B}(V, U) = 0$ for $i = 1, j = 2$.

So we conclude that the canonical block decomposition of a matrix corresponding to the direct sum decomposition $\mathbf{A} = K\mathbf{e} \oplus U \oplus V$, with β_1, β_2 K-bases of U and V respectively, is

$$[\mathfrak{B}] = \begin{array}{cc} \begin{array}{ccc} \mathbf{e} & \beta_1 & \beta_2 \end{array} & \\ \begin{pmatrix} \mathfrak{B}(\mathbf{e}, \mathbf{e}) & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix} & \begin{array}{c} \mathbf{e} \\ \beta_1 \\ \beta_2 \end{array} \end{array} \tag{14}$$

PROPOSITION 4. *If a T-algebra of rank 3, with an idempotent element, accepts a nondegenerate associative bilinear form, then it is a Jordan algebra.*

Proof. Let $(\mathbf{A}, \omega)$ be a T-algebra whose rank equation is (9), and let $\mathfrak{B}$ be an associative bilinear form in $\mathbf{A}$.

(i) If $\gamma = \frac{1}{2}$, we then have $n = 4\mathbf{e}n - 4\mathbf{e}(\mathbf{e}n)$, so for every $n \in N$ the identity $\mathfrak{B}(\mathbf{e}, n) = \mathfrak{B}(\mathbf{e}, 4\mathbf{e}n) - \mathfrak{B}(\mathbf{e}, 4\mathbf{e}(\mathbf{e}n)) = 4[\mathfrak{B}(\mathbf{e}, \mathbf{e}n) - \mathfrak{B}(\mathbf{e}^2, \mathbf{e}n)] = 0$ holds, i.e., $\mathfrak{B}(\mathbf{e}, N) = 0$. Additionally, for arbitrary $n_1, n_2 \in N$ we have

$$\begin{aligned}
\mathfrak{B}(n_1, n_2) &= \mathfrak{B}(4\mathbf{e}n_1 - 4\mathbf{e}(\mathbf{e}n_1), 4\mathbf{e}n_2 - 4\mathbf{e}(\mathbf{e}n_2)) \\
&= 16[\mathfrak{B}(\mathbf{e}n_1, \mathbf{e}n_2) - \mathfrak{B}(\mathbf{e}n_1, \mathbf{e}(\mathbf{e}n_2))] - \mathfrak{B}(\mathbf{e}(\mathbf{e}n_1), \mathbf{e}n_2) \\
&\quad + \mathfrak{B}(\mathbf{e}(\mathbf{e}n_1), \mathbf{e}(\mathbf{e}n_2)) \\
&= 0,
\end{aligned}$$

because $\mathfrak{B}$ is associative and N is a subalgebra of $\mathbf{A}$. Then

$$[\mathfrak{B}] = \begin{array}{c} \begin{array}{cc} \epsilon & \beta \end{array} \\ \begin{pmatrix} \mathfrak{B}(\mathbf{e}, \mathbf{e}) & 0 \\ 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \mathbf{e} \\ \beta, \end{array}$$

where β is any K-base of N. Therefore, if $\mathfrak{B}$ is not degenerate, then $N = \langle 0 \rangle$, from which $\mathbf{A} = K\mathbf{e}$ and $\mathbf{A}$ is a Jordan algebra.

(ii) If $\gamma \neq \frac{1}{2}$, we already have seen that the matrix canonical block form of $\mathfrak{B}$ is given by (14). When $0 \neq \gamma \neq 1$, using the Lemma in [1], we find that the matrix form of $\mathfrak{B}$ is

$$[\mathfrak{B}] = \begin{array}{c} \begin{array}{ccc} K\mathbf{e} & \beta_1 & \beta_2 \end{array} \\ \begin{pmatrix} \mathfrak{B}(\mathbf{e}, \mathbf{e}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \mathbf{e} \\ \beta_1 \\ \beta_2 \end{array}$$

That is, the bilinear form $\mathfrak{B}$ is degenerate except when $N = 0$. In this case $\mathbf{A} = K\mathbf{e}$ and $\mathbf{A}$ is a Jordan algebra. When $\gamma = 0$ or $\gamma = 1$, $\mathbf{A}$ is a Jordan algebra by Proposition 3. ■

PROPOSITION 5. *The associative bilinear form $\mathfrak{B}$ defined in a T-algebra of rank 3 with idempotent element is symmetric if and only if $\mathfrak{B}|_V$ is symmetric.*

Proof. When $\gamma \neq \frac{1}{2}$, we observe that if $v \in V$, $u_1, u_2 \in U$, then the identities $\mathfrak{B}(\mathbf{e}, v) = \mathfrak{B}(\mathbf{e}^2, v) = \mathfrak{B}(\mathbf{e}, \mathbf{e}v) = \mathfrak{B}(\mathbf{e}v, \mathbf{e}) = \mathfrak{B}(v, \mathbf{e}^2) = \mathfrak{B}(v, \mathbf{e})$ and $\mathfrak{B}(u_1, u_2) = 2\mathfrak{B}(\frac{1}{2}u_1, u_2) = 2\mathfrak{B}(\mathbf{e}u_1, u_2) = 2\mathfrak{B}(\mathbf{e}, u_1u_2) = 2\mathfrak{B}(\mathbf{e}u_2, u_1) = \mathfrak{B}(u_2, u_1)$ hold. Therefore the symmetry of $\mathfrak{B}$ only depends on the symmetry of $\mathfrak{B}|_V$.

When $\gamma = \frac{1}{2}$, we know, by the proof of Proposition 4, that $\mathfrak{B}(x, y) \neq 0$ only if $x = y = \mathbf{e}$. ∎

PROPOSITION 6. *Let* $(\mathbf{A}, \omega)$ *be a T-algebra with idempotent element* $\mathbf{e}$, $\mathbf{A} \neq K\mathbf{e}$, *and rank equation* $x^3 - \omega(x)x^2 = 0$. *Then the nondegenerate associative bilinear forms in* $\mathbf{A}$ *are given by the nondegenerate associative bilinear forms in a complementary vectorial subspace to* U^2 *in* V.

Proof. $\mathbf{A}$ is a T-algebra of rank 3, with $\gamma = 0$. Walcher [6] proves that in this case $\mathbf{A}$ is a Bernstein-Jordan algebra. Since $\mathbf{A}$ is a Bernstein algebra, we use the results established in [1] to complete the proof. ∎

PROPOSITION 7. *Let* $(\mathbf{A}, \omega)$ *be a T-algebra with idempotent element* $\mathbf{e}$, $\mathbf{A} \neq K\mathbf{e}$, *and rank equation* $x^3 - 2\omega(x)x^2 + \omega(x)^2x = 0$. *Then any associative bilinear form is totally determined by its values in* $(\mathbf{e}, \mathbf{e})$ *and* $(\mathbf{e}, V)$.

Proof. $\mathbf{A}$ is a T-algebra of rank 3 with $\gamma = 1$. Let $\mathfrak{B}$ be an associative bilinear form in $\mathbf{A}$; then the matrix canonical block form is given by (14). When $v_1, v_2 \in V$, we use $v_1v_2 \in V^2 = 0$ in order to prove $\mathfrak{B}(v_1, v_2) = 0$, i.e. $\mathfrak{B}(V, V) = 0$. Additionally, if $u_1, u_2 \in U$, we have $\mathfrak{B}(u_1, u_2) = 2\mathfrak{B}(\frac{1}{2}u_1, u_2) = 2\mathfrak{B}(\mathbf{e}, u_1u_2)$; in other words, the values which $\mathfrak{B}(U, U)$ takes are determined by the values of $\mathfrak{B}(\mathbf{e}, V)$. ∎

PROPOSITION 8. *Let* $\mathfrak{B}$ *be an associative bilinear form defined in a T-algebra of rank* 3. *Then* $\mathfrak{B}(\mathbf{e}_1, \mathbf{e}_1) = \mathfrak{B}(\mathbf{e}, \mathbf{e})$ *for any idempotent elements* $\mathbf{e}, \mathbf{e}_1$ *in* $\mathbf{A}$.

Proof. Let $(\mathbf{A}, \omega)$ be a T-algebra.

(i) Suppose that $\gamma = \frac{1}{2}$. Since $\mathbf{A} = K\mathbf{e} \oplus N$, $N = \ker \omega$, and $\mathbf{e}_1$ is of weight 1, we can write $\mathbf{e}_1 = \mathbf{e} + n_0$, giving $n_0 = 2\mathbf{e}n_0 + n_0^2$ and $\mathfrak{B}(n_0, n_0) = 4\mathfrak{B}(\mathbf{e}n_0, \mathbf{e}n_0) = 4\mathfrak{B}(\mathbf{e}, n_0(\mathbf{e}n_0))$. By (12) we have $n = 4\mathbf{e}n - 4\mathbf{e}(\mathbf{e}n)$ for every $n \in N$. Then $\mathfrak{B}(\mathbf{e}, n) = 0$, and consequently $\mathfrak{B}(n_0, n_0) = 0$. Now we easily obtain the desired conclusion.

(ii) Let $\gamma \neq \frac{1}{2}$. Costa has proved in [2] that in this case $u \in U$ exists such that $\mathbf{e}_1 = \mathbf{e} + u + (1 - 2\gamma)^{-1}u^2$. A direct calculation gives us

$$\mathfrak{B}(\mathbf{e}_1, \mathbf{e}_1) = \mathfrak{B}(\mathbf{e}, \mathbf{e}) + \frac{4(1-\gamma)}{1-2\gamma}\mathfrak{B}(\mathbf{e}, u^2).$$

There only remains the study of the case $\gamma \neq 1$. Since $\mathfrak{B}(\mathbf{e}, u^2) = \mathfrak{B}(\mathbf{e}, \mathbf{e}u^2) = \gamma\mathfrak{B}(\mathbf{e}, u^2)$, then $\mathfrak{B}(\mathbf{e}, u^2) = 0$. ■

REMARK. With the above hypothesis, when $u \in U$ satisfies $u^2 = 0$, every associative bilinear form in $\mathbf{A}$ is degenerate.

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